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BASICS OF CALCULUS

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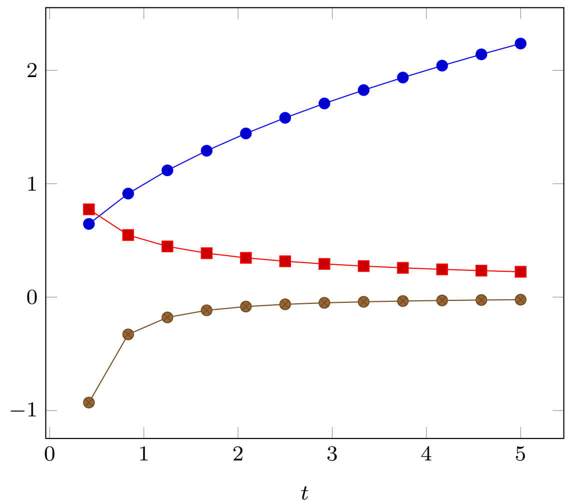

Figure 1. Blue circles represent position x of object over time t, according to law of motion $x = \sqrt{t}$. Red rectangles are velocity v of object over time t. Brown circles are acceleration $v' = -1/4x^{3/2}$.

1. DIFFERENTIATION

Calculus is the study of change. It is divided into two categories: Differential Calculus and Integral Calculus which are connected by the Fundamental Theorem of Calculus. Consider the law of motion given for some object as shown in figure 1:

$$x = \sqrt{t} \tag{1}$$

Velocity v which represents speed, or rate of change of x is a derivative of x over t:

$$\tan\alpha = v = \frac{\mathrm{d}x}{\mathrm{d}t} = x' = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \tag{2}$$

Note that $\tan\alpha$ provides geometric interpretation of a derivative as the angle of inclination of the tangent line to the curve at the point with abscissa. In order to determine velocity as derivative of x, one can *differentiate* the given law of motion:

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}x = (\sqrt{t})' = (t^{1/2})' = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}} \tag{3}$$

Thus, one can compute v for any t using this result. The rules of differentiation and table of derivatives are available from various sources, e.g. §6 of Chapter II "Analysis" in *Mathematics, its content, methods, and meaning* by Aleksandrov, Kolmogorov and Lavrent'ev. Note that second derivative $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = x'' = v'$ is acceleration.

2. INTEGRATION

On the other hand, the differential equation itself can come from empirical observations of the law of motion, astronomical observations for example:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2\sqrt{t}} \tag{4}$$

If one wishes to obtain position x, he can solve the differential equation 4 by *integrating*, i.e. taking "inverse" action of differentiation:

$$x = \int \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t \tag{5}$$

Area of graph (vt) provides position x. Because of the definition of integral as area (e.g. $\int_0^1 x^2\mathrm{d}x$ provides an area bounded by $y = x^2$ and abscissa within $0 \le x \le 1$), position is the limit of the sum of instantaneous velocities $v(\xi_i)$, multiplied by time intervals Δt_i, as those intervals approach zero:

$$x = \int v(t)\mathrm{d}t = \lim_{\max\Delta t_i \to 0} \sum_{i=1}^{n} v(\xi_i)\Delta t_i \tag{6}$$

Solving indefinite integral:

$$x = \int \frac{1}{2\sqrt{t}}\mathrm{d}t = \frac{1}{2}\int t^{-1/2}\mathrm{d}t = \frac{1}{2}2t^{0.5} + C = \sqrt{t} + C \tag{7}$$

Where C is constant that depends on initial conditions. The rules of integration are usually provided in sources similar to the one referenced above. With initial conditions $x_0 = 0$, $t_0 = 0$: $x_0 = \sqrt{t_0} + C$, thus $C = x_0 - \sqrt{t_0} = 0$, so $x = \sqrt{t}$, which returns us back to equation 1.

3. FORMULA OF NEWTON AND LEIBNITZ

The following Newton–Leibnitz equality is the corollary of the fundamental theorem of calculus that links definite integration of a function and its primitive (antiderivative):

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a) = F(x)\Big|_a^b \tag{8}$$

With primitive given as $F'(x) = f(x)$. For example, with primitive for x^2 being $x^3/3$:

$$\int_0^a x^2\mathrm{d}x = \frac{x^3}{3}\Big|_0^a = \frac{a^3}{3} - \frac{0}{3} = \frac{a^3}{3} \tag{9}$$

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